Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 14, 2003*
Revised: *June 12, 2003*
Accepted: *June 20, 2003*

Noncommutative probability, matrix models, and quantum orbifold geometry

Chi-Wei Herbert Lee

Department of Pure Mathematics, Faculty of Mathematics, University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada, and
Department of Physics, Faculty of Science, University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada
E-mail: h11lee@math.uwaterloo.ca

Abstract: Inspired by the intimate relationship between Voiculescu's noncommutative probability theory (of type A) and large-N matrix models in physics, we look for physical models related to noncommutative probability theory of type B. These turn out to be fermionic matrix-vector models at the double large-N limit. In the context of string theory, they describe different orbifolded string worldsheets with boundaries. Their critical exponents coincide with that of ordinary string worldsheets, but their renormalised tree-level one-boundary amplitudes differ.

Keywords: Models of Quantum Gravity, Matrix Models, 2D Gravity, Lattice Models of Gravity.

http://jhep.sissa.it/archive/papers/jhep062003044/jhep062003044.pdf

JHEP06(2003)044

Contents

1. Introduction **1**

2. Noncommutative probability and large-N matrix models **2**

3. A fermionic matrix-vector model **4**

4. Quantum orbifold geometry **8**

5. Another quantum orbifold geometry **12**

6. Conclusion and outlook **16**

1. Introduction

The interplay between noncommutative probability and large-N matrix models is a wonderful example of how mathematics and physics mutually benefit from each other. Noncommutative probability theory (of type A) was originally developed by Voiculescu as a tool to analyse some von Neumann algebras [1]; this analysis was rendered possible by random matrices. It can be shown that a large class of random matrices is asymptotically free as the order of the matrices is taken to infinity. Hence it constitutes a major example of noncommutative probability.

Nonetheless, random matrices show up in physics as well. An important example is Yang–Mills theory. The “infinite-dimensional” Yang–Mills matrix field was dubbed the master field [2]. The master field enjoys greatly simplifying features and is believed to hold the key to a deeper understanding of Yang–Mills theory. Among other examples are quantum gravity and string theory. The large-N expansion of matrix models was found out to be a genus expansion [3]; the dual of the Feynman diagrams of the leading term may be treated as discretised string worldsheets with the topology of a sphere [4]–[6]. The commonality of random matrices in mathematics and physics triggers an interest in the relationship between noncommutative probability theory of type A and large-N matrix models [7]–[12]. It is now possible to think of notions of noncommutative probability in physical terms and describe the master field in algebraic terms.

However, mathematicians have recently realised that other noncommutative probability spaces exist. This is achieved by a reformulation of noncommutative probability theory of type A in terms of non-crossing partitions of type A [13], which, in turn, is related to the Möbius inversion theory in the lattices of non-crossing partitions [14]. Then the fact that other types of non-crossing partitions exist [15] prompted an attempt to develop a “noncommutative probability theory of type B” from non-crossing partitions of type B [16].

Given the intimate relation between noncommutative probability theory of type A and large-N matrix models, naturally we would like to ask if there exist large-N matrix models "of type B".

Yes, such matrix models exist; they are fermionic matrix-vector models in the double large-N limit. Fermionic matrix (vector) models [17] are models in which the matrix entries (vector components) are Grassmann numbers. Their convergent behavior is, in general, better than their bosonic counterparts. Physically, they describe two-dimensional quantum gravity, random polymers [18]–[20], or induced gauge theory [21]. The basic ingredients of the models in this article are N_v-dimensional vectors of $N_m \times N_m$ matrices of Grassmann numbers; these are thus simultaneously matrices and vectors. We will take N_v to infinity first and N_m to infinity afterwards. Then the dual of the Feynman diagrams of these models describes orbifolded string worldsheets in string theory.

We will deal with the physical aspect of fermionic matrix-vector models in this article in detail; the more mathematical aspect will be discussed in future works.

Here is a synopsis of this article. We will review the relationship between noncommutative probability theory of type A and large-N matrix models in section 2. Then we will introduce non-crossing partitions of type B and a class of fermionic matrix-vector models which generates these partitions in section 3. In section 4, we will identify these partitions as random surfaces of a bounded region of the orbifold $\mathbb{R}^2/\mathbb{Z}_2 \times \mathbb{Z}_2$ and compute the full Green functions (moments) at the double large-N limit. We will also show by means of the Schwinger–Dyson equation that the renormalised tree-level one-boundary amplitude and critical exponent associated with the string susceptibility at the continuum limit are identical to those of ordinary quantum gravity. In Section 5, we will construct a different class of fermionic matrix-vector models which generates random surfaces of a bounded region of the quantum orbifold $\mathbb{R}^2/\mathbb{Z}_2$. We will show that its critical exponent is also the same as that of ordinary quantum gravity, but it has a different one-boundary amplitude. Finally, we will indicate possible extensions of this work in section 6.

2. Noncommutative probability and large-N matrix models

A *noncommutative probability space* (of type A) was originally defined by D. Voiculescu [1] as an ordered pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a complex unital C*-algebra and $\varphi : \mathcal{A} \to \mathbf{C}$ a positive linear functional satisfying the normalisation condition that $\varphi(1) = 1$. Each element a of $\mathcal{A}$ may be considered as a random variable and $\varphi(a)$ its expectation value.

A useful notion of noncommutative probability theory is the non-crossing cumulant [13]. To define what it is, let us digress for a moment and introduce a few auxiliary notions first. Consider a finite set F of integers. Let p be a partition of F. We write $n_1 \simeq_p n_2$ if n_1 and n_2 are numbers of F such that they are in the same block (or cell) of the partition p. Furthermore, p is a *non-crossing partition of type A* if $n_1 \simeq_p n_3$ and $n_2 \simeq_p n_4$ imply that $n_2 \simeq n_3$ for any four integers n_1, n_2, n_3, and n_4 in F such that $n_1 < n_2 < n_3 < n_4$. An example of a non-crossing partition of type A is illustrated in figure 1. The set of all non-crossing partitions of type A of F is denoted as $NC^{(A)}(F)$. If $F = \{1, 2, \ldots, n\}$, we will abbreviate $NC^{(A)}(F)$ as $NC^{(A)}(n)$ as there will be no danger of confusion.

JHEP06(2003)044

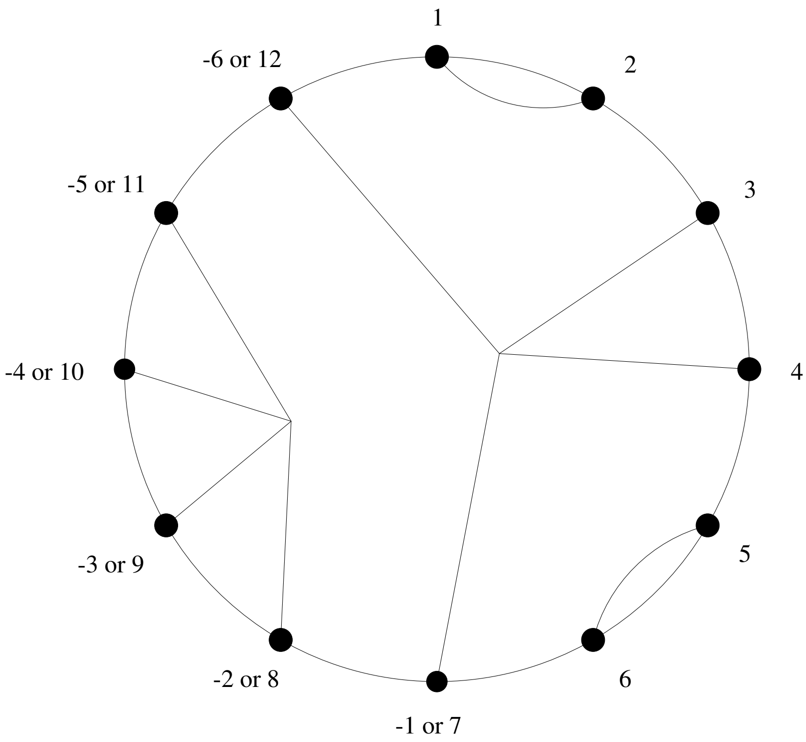


Figure 1: A non-crossing partition of type A but not of type B. The partition of the numbers 1, 2, ..., 12 into the blocks $\{1,2\}$, $\{3,4,7,12\}$, $\{5,6\}$, and $\{8,9,10,11\}$ is the non-crossing partition of type A, but the (same) partition of the numbers 1, 2, ..., 6, -1, -2, ..., -6 into the blocks $\{1,2\}$, $\{3,4,-1,-6\}$, $\{5,6\}$, and $\{-2,-3,-4,-5\}$ is not a non-crossing partition of type B. This may also be regarded as a planar Feynman diagram with each intersection point and the lines connected to it inside the great circle representing a connected Green function.

A *non-crossing cumulant* is a multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbf{C}$, where n is any positive integer, such that

$$\varphi(a_1 a_2 \cdots a_k) = \sum_{p \in NC^{(A)}(k)} \prod_{F \text{ is a block of } p} \kappa_{\text{card}(F)}((a_1, a_2, \ldots, a_k)|F) \,. \tag{2.1}$$

In this formula, if $F = \{j_1, j_2, \ldots, j_m\}$ such that $j_1 < j_2 < \ldots < j_m$, then card$F$, the cardinality of F, is m, and

$$(a_1, a_2, \ldots, a_k)|F := (a_{j_1}, a_{j_2}, \ldots, a_{j_m}) \in \mathcal{A}^m \,.$$

We may apply eq. (2.1) recursively to obtain the expressions of all non-crossing cumulants in terms of φ. For example,

$$\begin{aligned} \kappa_1(a_1) &= \varphi(a_1)\,, \\ \kappa_2(a_1,a_2) &= \varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)\,, \\ \kappa_3(a_1,a_2,a_3) &= \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_2)\varphi(a_1a_3) - \varphi(a_3)\varphi(a_1a_2) + \\ &\quad + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)\,, \end{aligned} \tag{2.2}$$

and so on. The significance of non-crossing cumulants lies in the fact that they provide a particularly simple criterion to determine whether a family of unital subalgebras of $\mathcal{A}$ is free and that the generating series of the non-crossing cumulants, called the R-transform, display simplifying properties in freeness computations [1].

Random matrix models in the large-N limit furnish a nice physical example of non-commutative probability theory of type A [8]. In the simplest case, the random variables are, intuitively speaking, generated by a hermitean random matrix M of infinite order, which is alternatively known as the master field. The partition function Z_b of this random matrix model takes the form

$$Z_b = \int dM \exp(-N \operatorname{Tr} V(M)),$$

where N is the order of the matrix M and will eventually be taken to infinity, $V(M)$ is a polynomial of M, and dM is the corresponding measure. The expectation value of the random variable M^n is

$$\tilde{\phi}(n/2) := \lim_{N\to\infty} \frac{1}{Z_b} \int dM \frac{1}{N} \operatorname{Tr} M^n \exp(-N \operatorname{Tr} V(M)). \tag{2.3}$$

$\tilde{\phi}(n/2)$ is called a *full Green function* in physics literature. The non-crossing cumulant $\kappa_n(M, M, \ldots, M)$ is nothing but the connected Green function

$$\lim_{N\to\infty} \frac{1}{N} \langle \operatorname{Tr} M^n \rangle_c ,$$

and a non-crossing partition of type A may be represented by a planar Feynman diagram (figure 1). Formulae (2.2) are the expressions of connected Green functions in terms of full Green functions familiar to physicists.

3. A fermionic matrix-vector model

There are other types of non-crossing partitions which are of interest in mathematics [15]. For instance, consider the totally ordered set

$$[\pm n] := \{1 < 2 < \cdots < n < -1 < -2 < \cdots < -n\}.$$

It is obviously isomorphic to the set

$$[2n] := \{1 < 2 < \cdots 2n\}.$$

Moroever, define the *inversion map* to be a map which maps the integer a to $-a$. *A non-crossing partition of type B* of $[\pm n]$ is a non-crossing partition of type A of $[2n]$ such that the partition is invariant under the inversion map. Not all non-crossing partitions of type A are of type B as well; this can be seen in figure 1. The set of all non-crossing partitions of type B of $[\pm n]$ is denoted as $NC^{(B)}(n)$. A cell of a non-crossing partition of type B is called a *zero-block* if it is invariant under the inversion map; otherwise, it is called

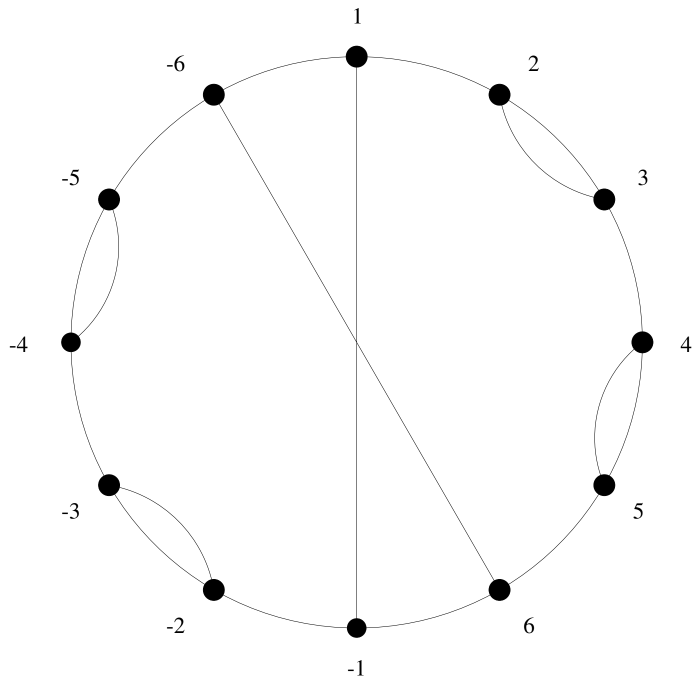


Figure 2: A non-crossing partition of type B. The partition of the numbers 1, 2, . . . , 6, -1, -2, . . . , -6 into the blocks $\{1, 6, -1, -6\}$, $\{2, 3\}$, $\{4, 5\}$, $\{-2, -3\}$, and $\{-4, -5\}$ is a non-crossing partition of type B. $\{1, 6, -1, -6\}$ is a zero-block, and all others are non-zero-blocks.

a *non-zero-block*. It can be straightforwardly shown that there is at most one zero-block in every non-crossing partition of type B. A partition with a zero-block is illustrated in figure 2.

Associated with non-crossing partitions of type B is a nascent development of a noncommutative probability theory of type B [16] in which the non-crossing cumulants are defined in terms of non-crossing partitions of type B. Given the intimate relationship between noncommutative probability theory and matrix models, it is worthwhile to search for matrix models which are physical realisation of noncommutative probability theory of type B. It turns out that there is such a class of matrix models — the fermionic matrix-vector models in the double large-N limit.

A fermionic matrix-vector model is built out of an N_v-dimensional vector of matrices Ψ_1, Ψ_2, . . . , and Ψ_{N_v}, and another N_v-dimensional vector of matrices $\bar{\Psi}_1$, $\bar{\Psi}_2$, . . . , and $\bar{\Psi}_{N_v}$. Each of these matrices is of order N_m. Every matrix entry is a Grassmann number. The i, j-th matrix entry of the matrices Ψ_μ and $\bar{\Psi}_\mu$ are denoted by $\psi_{\mu ij}$ and $\bar{\psi}_{\mu ij}$, respectively. Any two Grassmann numbers anti-commute with each other.

The action of a family of fermionic matrix-vector models takes the form

$$\begin{aligned} S := N_m \sqrt{N_v} \sum_{\mu=1}^{N_v} \mathrm{Tr}\, \bar{\Psi}_\mu \Psi_\mu + N_m \sum_{n=1}^{\infty} \frac{c_n}{2n} \sum_{\mu_1, \mu_2, \ldots, \mu_{2n}=1}^{N_v} \mathrm{Tr} \Big[\left(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}} \right)^2 \Big] + \\ + N_m^2 \sum_{n=2}^{\infty} \frac{g_n}{2n} \sum_{\mu_1, \mu_2, \ldots, \mu_{2n}=1}^{N_v} \left[\mathrm{Tr} \left(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \cdots \bar{\Psi}_{\mu_{2n-1}} \Psi_{\mu_{2n}} \right) \right]^2 , \end{aligned} \tag{3.1}$$

where c_1, c_2, c_3, ..., and so on and g_2, g_3, g_4, ..., and so on are constant complex numbers. The partition function is given by

$$Z_{N_m,N_v} := \int d\Psi_1 d\bar{\Psi}_1 d\Psi_2 d\bar{\Psi}_2 \cdots d\Psi_{N_v} d\bar{\Psi}_{N_v} \exp S\,, \tag{3.2}$$

where the Grassmann integrals are defined as usual in the sense of Brezin [22]. The full Green functions which are of interest to us take the form

$$G_{N_m,N_v}(n) := \frac{1}{N_m} \sum_{\mu_1,\mu_2,\ldots,\mu_{2n}=1}^{N_v} \left\langle \mathrm{Tr}\left[\left(\bar{\Psi}_{\mu_1}\Psi_{\mu_2}\bar{\Psi}_{\mu_3}\Psi_{\mu_4}\cdots\bar{\Psi}_{\mu_{2n-1}}\Psi_{\mu_{2n}}\right)^2\right]\right\rangle_S\,,$$

where the subscript S means that this expectation value is evaluated with respect to the partition function defined by the action S.

The Feynman rules are analogous to those of U(N) gauge theory [3]. Some examples are depicted in figure 3. Note that a *double-end*, i.e., a pair of ends of a double-line representing a propagator or a double-leg of a vertex, represents a row index of an $N_m \times N_m$ matrix, a column index of the same matrix and an index of an N_v-dimensional vector. If the subscript of a vector index is odd, then the corresponding double-end originates from $\bar{\Psi}$; otherwise, it originates from Ψ. For instance, the indices μ_1 and μ_3 in the c_1-vertex originates from $\bar{\Psi}$, whereas the indices μ_2 and μ_4 originates from Ψ. As a result, we may connect two double-ends together by a propagator only if the subscript of the vector index of one of the double-ends is odd and the subscript of the vector index of the other double-end is even. For example, we may use a propagator to connect the double-end of a c_1 vertex whose vector index is μ_1 to the double-end of a g_2 vertex whose vector index is ν_4, but we may not use a propagator to connect the same double-end of the c_1 vertex with the double-end of the g_2 vertex whose vector index is ν_3. Each of the lines connects a row index to a column index; the connections have nothing to do with vector indices. A fermionic loop may contribute a factor of N_m or $-N_m$. The sign is determined by first principles; two Feynman diagrams which are topologically equivalent after removal of the vector indices may carry opposite signs.

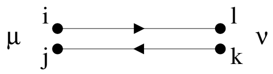
The *double large-N limit* is characterised by the following formulae for the partition function and the Green functions:

$$\begin{aligned} Z &:= \lim_{N_m\to\infty}\lim_{N_v\to\infty} Z_{N_m,N_v}\,, \qquad \text{and} \\ G(n) &:= \lim_{N_m\to\infty}\lim_{N_v\to\infty} G_{N_m,N_v}(n)\,. \end{aligned} \tag{3.3}$$

Note that we take N_v to infinity *before* taking N_m to infinity; this ensures that only connected Feynman diagrams will contribute to $G(n)$. Moreover, only those Feynman diagrams which satisfy the following condition survive the large-N_v limit:

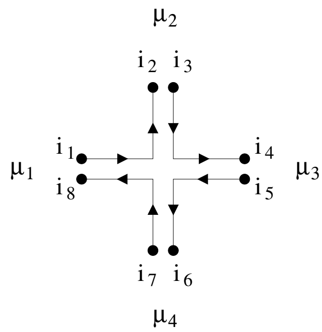
> *if, in a Feynman diagram, a propagator connects together two double-ends whose vector indices are μ_a and ν_b, then in the same diagram there must be another propagator connecting two other double-ends whose vector indices are μ_a and ν_b, too.*

JHEP06(2003)044



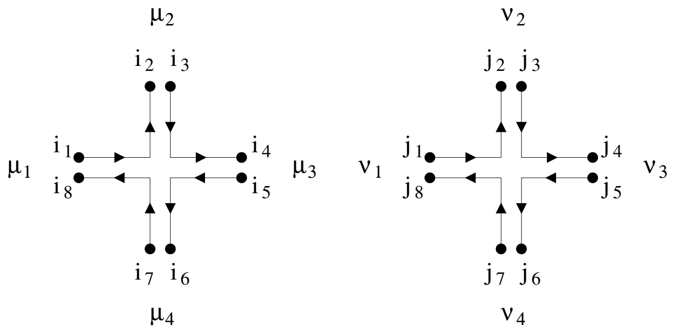
$\pm\,\delta_{\mu\nu}\ \delta_{il}\ \delta_{kj}$

(a) propagator



$$\delta_{\mu_1\mu_3}\,\delta_{\mu_2\mu_4}\,\delta_{i_1i_2}\,\delta_{i_3i_4}\,\delta_{i_5i_6}\,\delta_{i_7i_8}\,\frac{c_1}{2}$$

(b) c_1- vertex



$$\delta_{\mu_1\nu_1}\,\delta_{\mu_2\nu_2}\,\delta_{\mu_3\nu_3}\,\delta_{\mu_4\nu_4}\,\delta_{i_1i_2}\,\delta_{i_3i_4}\,\delta_{i_5i_6}\,\delta_{i_7i_8}\,\delta_{j_1j_2}\,\delta_{j_3j_4}\,\delta_{j_5j_6}\,\delta_{j_7j_8}\,\frac{g_2}{4}$$

(c) g_2- double-vertex

Figure 3: Selected Feynman rules for a fermionic matrix-vector model. Depicted here are (a) a propagator, (b) a vertex with a coefficient c_1, and (c) a double-vertex with a coefficient g_2. Each pair of solid circles represents a double-end. Each fermionic loop contributes, up to a sign, a factor of N_m.

Furthermore, only planar Feynman diagrams which survive the large-N_v limit contribute to the double large-N limit. An example of such a Feynman diagram is illustrated in figure 4. This diagram contributes to figure 2, which, as a Feynman diagram, each intersection point and the lines connected to it inside the great circle represent a connected Green function, any pair of vector indices μ_k of the sources are abbreviated as k and $-k$ for any value of k, and the vector indices of the vertices are suppressed.

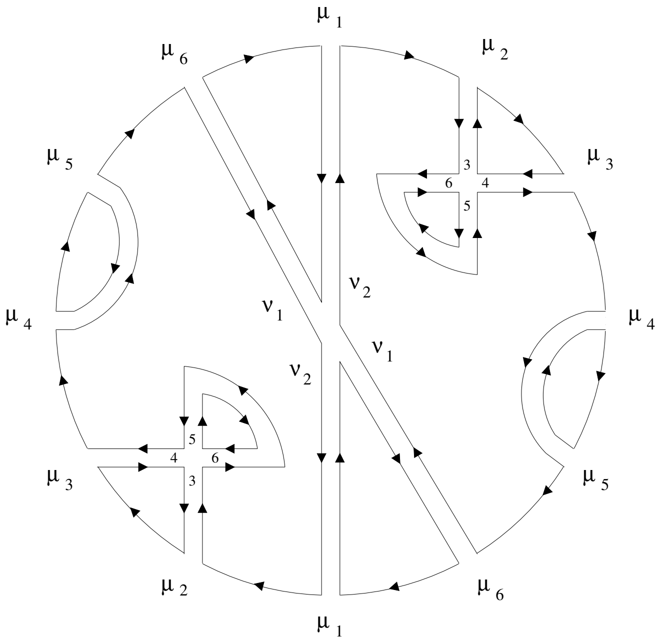


Figure 4: A Feynman diagram of $G(3)$. Only vector indices are shown. The numbers 3, 4, 5, and 6 stand for ν_3, ν_4, ν_5, and ν_6, respectively.

4. Quantum orbifold geometry

It is well known that the dual of the Feynman diagrams of matrix models associated with noncommutative probability theory of type A may be regarded as discrete random surfaces of two-dimensional quantum gravity. (See, e.g, refs. [23] and [24] and the references therein.) Does the fermionic matrix-vector model play a role in quantum gravity as well?

A clue may be found in figure 4. Choose the center of the great circle to be the origin of a two-dimensional cartesian coordinate system. Point the y-axis to the index μ_1. Then the figure respects parity transformation $x \to -x$ and $y \to -y$ about the origin. So does its dual. Though a thoroughly rigorous proof is lacking, it is probably true that every Feynman diagram of $G(n)$ which contributes to the double large-N limit exhibits the same symmetry. Hence we assert that *the dual of the Feynman diagrams of $G(n)$ in the double large-N limit may be thought of as quadrangulated random surfaces which respect parity transformation about the origin, i.e., these are random surfaces of a bounded region of the orbifold $\mathbb{R}^2/\mathbb{Z}_2 \times \mathbb{Z}_2$.* (c.f. figure 5.)

Let us study the full Green functions and critical behavior of the simplest fermionic matrix-vector model in which only the coefficients $c = c_1$ and $g = g_2$ are non-zero. From eq. (3.1), the action is

$$S := N_m \sqrt{N_v} \sum_{\mu=1}^{N_v} \mathrm{Tr}\, \bar{\Psi}_\mu \Psi_\mu + \frac{N_m c}{2} \sum_{\mu_1,\mu_2=1}^{N_v} \mathrm{Tr} \left[\left(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \right)^2 \right] + \\ + \frac{N_m^2 g}{4} \sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{N_v} \left[\mathrm{Tr} \left(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \right) \right]^2 . \tag{4.1}$$

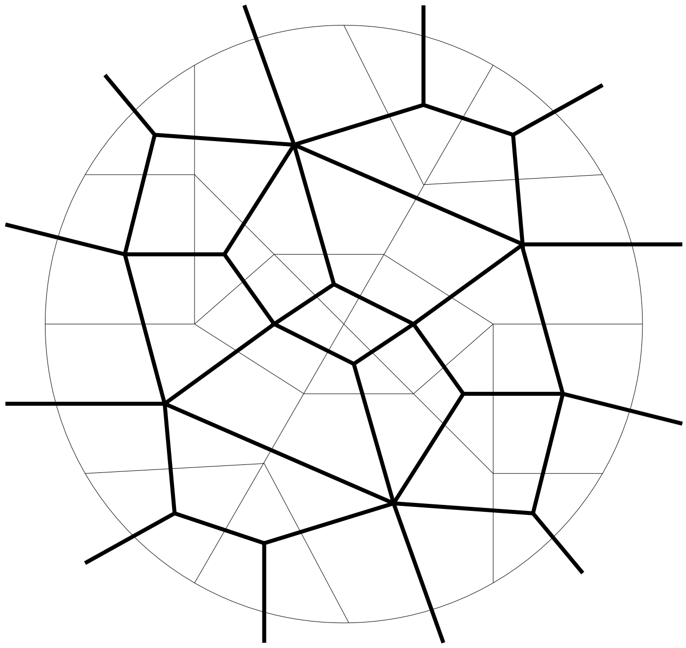


Figure 5: A discretized random orbifold surface with a boundary. The quadrilaterals paving the surface are in thick lines, and the thin circle forms the boundary. The dual of this surface, drawn in thin lines, is a Feynman diagram of the fermionic matrix-vector model defined in eq. (4.1) in the double large-N limit. Note that the "spokes" protruding from the great circle converge to the same point on a spherical surface.

The full Green functions can be obtained by the Schwinger–Dyson approach. Indeed, consider the trivial equation

$$\lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^2\sqrt{N_v}Z_{N_m,N_v}}\sum_{i,j=1}^{N_m}\sum_{\beta=1}^{N_v}\int d\Psi_1 d\bar{\Psi}_1 d\Psi_2 d\bar{\Psi}_2\cdots d\Psi_{N_v}d\bar{\Psi}_{N_v}\times$$
$$\times\frac{\partial}{\partial\bar{\Psi}_{\beta ij}}\left\{\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}=1}^{N_v}\left(\bar{\Psi}_{\alpha_1}\Psi_{\alpha_2}\cdots\bar{\Psi}_{\alpha_{2n-1}}\Psi_\beta\bar{\Psi}_{\alpha_1}\Psi_{\alpha_2}\cdots\bar{\Psi}_{\alpha_{2n-1}}\right)_{ij}\exp S\right\} = 0\,, \quad (4.2)$$

where S was given in eq. (4.1) and n is any positive integer. Since

$$\lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^2\sqrt{N_v}}\sum_{i,j=1}^{N_m}\sum_{\beta=1}^{N_v}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}}^{N_v}\Bigg\langle\Bigg(\bar{\Psi}_{\alpha_1}\Psi_{\alpha_2}\cdots\Psi_{\alpha_{2k}}\times$$
$$\times\left(\frac{\partial}{\partial\bar{\Psi}_{\beta ij}}\bar{\Psi}_{\alpha_{2k+1}}\right)\Psi_{\alpha_{2k+2}}\cdots\bar{\Psi}_{\alpha_{2n-1}}\Psi_\beta\bar{\Psi}_{\alpha_1}\Psi_{\alpha_2}\cdots\bar{\Psi}_{\alpha_{2n-1}}\Bigg)_{ij}\Bigg\rangle_S = \tilde{\phi}(k)G(n-k-1),$$

where $\tilde{\phi}(n)$ was defined in eq. (2.3) with

$$V(M)=\frac{1}{2}M^2-\frac{g}{4}M^4$$

and $0 \leq k \leq n-2$,

$$\lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^2\sqrt{N_v}}\sum_{i,j=1}^{N_m}\sum_{\beta=1}^{N_v}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}}^{N_v}\left\langle\left(\bar\Psi_{\alpha_1}\Psi_{\alpha_2}\cdots\Psi_{\alpha_{2n-2}}\times\right.\right.$$
$$\left.\left.\times\left(\frac{\partial}{\partial\bar\Psi_{\beta ij}}\bar\Psi_{\alpha_{2n-1}}\right)\Psi_\beta\bar\Psi_{\alpha_1}\Psi_{\alpha_2}\cdots\bar\Psi_{\alpha_{2n-1}}\right)_{ij}\right\rangle_S = -\tilde\phi(n-1)\,,$$

$$\lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^2\sqrt{N_v}}\sum_{i,j=1}^{N_m}\sum_{\beta=1}^{N_v}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}}^{N_v}\left\langle\left(\bar\Psi_{\alpha_1}\Psi_{\alpha_2}\cdots\bar\Psi_{\alpha_{2n-1}}\Psi_\beta\times\right.\right.$$
$$\left.\left.\times\left(\frac{\partial}{\partial\bar\Psi_{\beta ij}}\bar\Psi_{\alpha_1}\right)\Psi_{\alpha_2}\cdots\bar\Psi_{\alpha_{2n-1}}\right)_{ij}\right\rangle_S = \tilde\phi(n-1)\,,$$

and

$$\lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^2\sqrt{N_v}}\sum_{i,j=1}^{N_m}\sum_{\beta=1}^{N_v}\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}}^{N_v}\left\langle\left(\bar\Psi_{\alpha_1}\Psi_{\alpha_2}\cdots\bar\Psi_{\alpha_{2n-1}}\Psi_\beta\bar\Psi_{\alpha_1}\Psi_{\alpha_2}\cdots\Psi_{\alpha_{2k}}\times\right.\right.$$
$$\left.\left.\times\left(\frac{\partial}{\partial\bar\Psi_{\beta ij}}\bar\Psi_{\alpha_{2k+1}}\right)\bar\Psi_{\alpha_{2k+2}}\cdots\bar\Psi_{\alpha_{2n-1}}\right)_{ij}\right\rangle_S =$$
$$= \tilde\phi(n-k-1)G(k)$$

for $1 \leq k \leq n-1$, it follows from eq. (4.2) that the *Schwinger–Dyson equations* are given by

$$-G(1) + c\tilde\phi(1) + gG(2) = 0 \tag{4.3}$$

and

$$2\sum_{k=0}^{n-2}\tilde\phi(k)G(n-1-k) - G(n) + c\tilde\phi(n) + gG(n+1) = 0 \tag{4.4}$$

for $n = 2, 3, 4, \ldots$, and so on.

Let

$$\phi(z) := \sum_{n=0}^{\infty}\tilde\phi(n)z^{2n}$$

be a generating function of the full Green functions of the bosonic matrix ϕ^4-theory. It is well known [25] that

$$\phi(z) = \frac{1}{2z^2} - \frac{g}{2z^4} - \frac{1}{z^2}\left(\frac{1}{2} - g\gamma^2 - \frac{g}{2z^2}\right)\sqrt{1-4\gamma^2z^2}\,, \tag{4.5}$$

where

$$\gamma^2 := \frac{1-\sqrt{1-12g}}{6g}\,. \tag{4.6}$$

Let

$$\omega_1(z) := \sum_{n=0}^{\infty}G(n)z^{2n}$$

be a generating function of the full Green functions of the fermionic matrix-vector model. It then follows from eqs. (4.3) and (4.4) that

$$\omega_1(z) = 1 + \frac{gG(1) + c - c\phi(z)}{[g - (1 - 2g\gamma^2)z^2]\sqrt{1 - 4\gamma^2 z^2}} z^2 \,. \tag{4.7}$$

It is clear from eq. (4.5) that $\phi(z)$ has a branch cut between $-1/(2\gamma)$ and $1/(2\gamma)$ but is otherwise holomorphic on the complex z-plane. The usual practice of matrix model analysis is to demand that the holomorphic structure of the generating function be as simple as possible [26]–[28] and [9, 17, 20]. Moving along the same spirit, we assert that $\omega_1(z)$ is holomorphic on the whole complex plane except the same branch cut. In particular, the singularities at

$$z = \pm\sqrt{\frac{g}{1 - 2g\gamma^2}}$$

are removable. Then the numerator of the second term of the right hand side of eq. (4.7) vanishes if $z^2 = g/(1 - 2g\gamma^2)$. This fixes $G(1)$:

$$G(1) = \frac{c}{g}\phi\left(\sqrt{\frac{g}{1 - 2g\gamma^2}}\right) - \frac{c}{g} \,. \tag{4.8}$$

Let

$$\omega(z) := \frac{1}{z}\omega_1(\frac{1}{z}) \,.$$

Then eqs. (4.7) and (4.8) yield

$$\omega(z) = 1 - \frac{cz^2}{2} + \rho(z) \,, \tag{4.9}$$

where

$$\rho(z) := \frac{cz\left[2(\gamma^2 + 2) - 3z^2 + 3gz^4\right]}{6\,(gz^2 - 1 + 2g\gamma^2)\sqrt{z^2 - 4\gamma^2}} \tag{4.10}$$

has a branch cut between -2γ and 2γ. It is clear from eq. (4.10) that $G(n) = 0$ for any positive value of n if $c = 0$. This is consistent with the well-known fact in fermionic matrix models that all even moments vanish if the model respects chiral transformation [17]

$$\Psi \to \bar{\Psi} \qquad \text{and} \qquad \bar{\Psi} \to -\Psi \,.$$

Criticality occurs if the zeros of $\rho(z)$ coalesce with an end-point of the branch cut. It then follows from eqs. (4.10) and (4.6) that the critical values of g and z are

$$g_* = 1/12 \text{ and } z_* = 2\sqrt{2} \,,$$

respectively. Set

$$g = g_* \exp(-a^2\Lambda_s) \qquad \text{and} \qquad z = z_* \exp(a\Lambda_l) \,,$$

where a is the *cut-off length* and Λ_s and Λ_l are the *continuum bulk and boundary cosmological constants*, respectively. (c.f. ref. [27].) At the *continuum limit*, a is small and eq. (4.9) becomes

$$\omega(z) \simeq 2ca^{-1/2} w(\Lambda_s, \Lambda_l) \,,$$

where

$$w(\Lambda_s,\Lambda_l) := \frac{1}{\left(\sqrt{\Lambda_s}+2\Lambda_l\right)^{1/2}}$$

may be regarded as the *renormalized tree-level one-boundary amplitude* of the orbifolded string worldsheet. This is different from the corresponding amplitude of ordinary quantum gravity. The scaling behavior of this fermionic matrix-vector model near criticality can be determined as usual by the *string susceptibility*

$$\begin{aligned}
\chi &:= \lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^4}\frac{\partial^2}{\partial g^2}\langle\log Z_{N_m,N_v}\rangle_S && (4.11)\\
&= \lim_{N_m\to\infty}\lim_{N_v\to\infty}\frac{1}{N_m^2}\frac{\partial}{\partial g}\left\langle\frac{1}{4}\sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{N_v}\left[\mathrm{Tr}\left(\bar\Psi_{\mu_1}\Psi_{\mu_2}\bar\Psi_{\mu_3}\Psi_{\mu_4}\right)\right]^2\right\rangle_S && (4.12)\\
&= \frac{\partial}{\partial g}\left(\tilde\phi(2)\right). && (4.13)
\end{aligned}$$

The *critical exponent* $\gamma_{\rm str}$ is defined by the formula

$$\chi \simeq (g-g_*)^{-\gamma_{\rm str}}$$

up to a proportionality constant which depends on g_* only. It then follows from Eqs.(4.13), (4.5), and (4.6) that

$$\gamma_{\rm str} = -\frac{1}{2},$$

just like that of ordinary quantum gravity without matter [23].

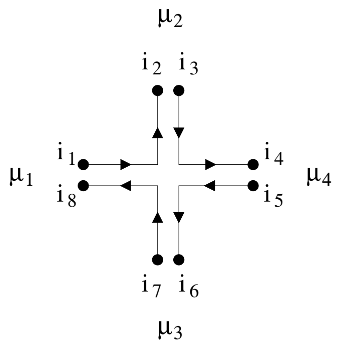
5. Another quantum orbifold geometry

We may arrange the fermionic matrix-vectors differently to form different models. For instance, consider another family of fermionic matrix-vector models whose actions take the form

$$\begin{aligned}
S' := {}& N_m\sqrt{N_v}\sum_{\mu=1}^{N_v}\mathrm{Tr}\,\bar\Psi_\mu\Psi_\mu + N_m\sum_{n=1}^{\infty}c'_n\sum_{\mu_1,\mu_2,\ldots,\mu_{2n}=1}^{N_v}\mathrm{Tr}\Big(\bar\Psi_{\mu_1}\Psi_{\mu_2}\bar\Psi_{\mu_3}\Psi_{\mu_4}\cdots\bar\Psi_{\mu_{2n-1}}\Psi_{\mu_{2n}}\times\\
&\times\Psi_{\mu_{2n}}\bar\Psi_{\mu_{2n-1}}\Psi_{\mu_{2n-2}}\bar\Psi_{\mu_{2n-3}}\cdots\Psi_{\mu_2}\bar\Psi_{\mu_1}\Big)+\\
&+N_m^2\sum_{n=2}^{\infty}\frac{g'_n}{2n}\sum_{\mu_1,\mu_2,\ldots,\mu_{2n}=1}^{N_v}\mathrm{Tr}\left(\bar\Psi_{\mu_1}\Psi_{\mu_2}\bar\Psi_{\mu_3}\Psi_{\mu_4}\cdots\bar\Psi_{\mu_{2n-1}}\Psi_{\mu_{2n}}\right)\times\\
&\times\mathrm{Tr}\left(\Psi_{\mu_{2n}}\bar\Psi_{\mu_{2n-1}}\Psi_{\mu_{2n-2}}\bar\Psi_{\mu_{2n-3}}\cdots\Psi_{\mu_2}\bar\Psi_{\mu_1}\right). \qquad (5.1)
\end{aligned}$$

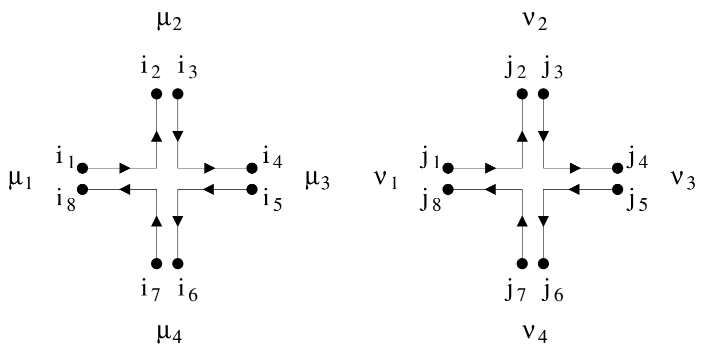
The partition function Z'_{N_m,N_v} is given by eq. (3.2) with S replaced with S'. We are interested in the full Green functions which read

$$\begin{aligned}
G'_{N_m,N_v}(n) := \frac{1}{N_m}\sum_{\mu_1,\mu_2,\ldots,\mu_{2n}=1}^{N_v}&\langle\mathrm{Tr}(\bar\Psi_{\mu_1}\Psi_{\mu_2}\bar\Psi_{\mu_3}\Psi_{\mu_4}\cdots\bar\Psi_{\mu_{2n-1}}\Psi_{\mu_{2n}}\times\\
&\times\Psi_{\mu_{2n}}\bar\Psi_{\mu_{2n-1}}\Psi_{\mu_{2n-2}}\bar\Psi_{\mu_{2n-3}}\cdots\Psi_{\mu_2}\bar\Psi_{\mu_1})\rangle_{S'}.
\end{aligned}$$



$$\delta_{\mu_1\mu_3}\,\delta_{\mu_2\mu_4}\,\delta_{i_1 i_2}\,\delta_{i_3 i_4}\,\delta_{i_5 i_6}\,\delta_{i_7 i_8}\,c'_1$$

(a) c'_1 - vertex



$$\delta_{\mu_1\nu_1}\,\delta_{\mu_2\nu_4}\,\delta_{\mu_3\nu_3}\,\delta_{\mu_4\nu_2}\,\delta_{i_1 i_2}\,\delta_{i_3 i_4}\,\delta_{i_5 i_6}\,\delta_{i_7 i_8}\,\delta_{j_1 j_2}\,\delta_{j_3 j_4}\,\delta_{j_5 j_6}\,\delta_{j_7 j_8}\,\frac{g'_2}{4}$$

(b) g'_2 - double-vertex

Figure 6: Selected vertices of the fermionic matrix-vector model defined in eq. (5.1). Depicted are (a) a vertex with a coefficient c'_1 and (b) a double-vertex with a coefficient g'_2.

The only changes to the Feynman rules are in the vertices. Some vertices are depicted in figure 6.

The double large-N limit is taken as in eq. (3.3) with Z, Z_{N_m,N_v}, $G(n)$, and $G_{N_m,N_v}(n)$ replaced with their primed versions. A Feynman diagram which survives the double large-N limit is illustrated in figure 7. If we choose the center of the great circle to be the origin of a two-dimensional cartesian coordinate system and point the y-axis to μ_1, then the diagram together with its dual respects reflection about the y-axis. This suggests that *the dual of the Feynman diagrams of $G'(n)$ in the double large-N limit may be conceived of as discrete random surfaces which respect reflection about the y-axis, i.e., these are random surfaces of a bounded region of the orbifold $\mathbb{R}^2/\mathbb{Z}_2$.* (c.f. figure 8.)

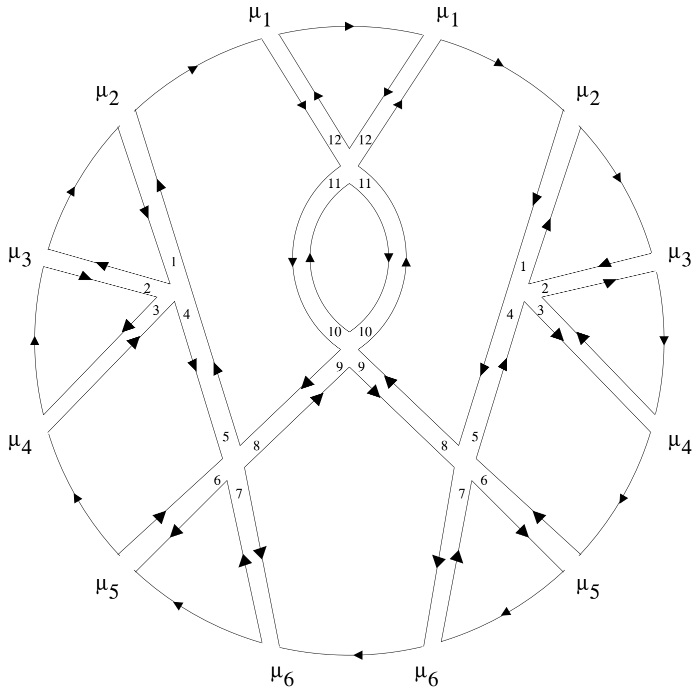


Figure 7: A Feynman diagram of $G'(3)$. Only vector indices are shown. The numbers 1, 2, 3, …, and so on stand for ν_1, ν_2, ν_3, and so on, respectively.

Let us concentrate on the simplest model in this family and choose only the coefficients $c = c'_1$ and $g = g'_2$ to be non-zero. It follows from eq. (5.1) that

$$S' := N_m \sqrt{N_v} \sum_{\mu=1}^{N_v} \operatorname{Tr} \bar{\Psi}_\mu \Psi_\mu + N_m c \sum_{\mu_1,\mu_2=1}^{N_v} \operatorname{Tr} \left(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \Psi_{\mu_2} \bar{\Psi}_{\mu_1} \right) + \\ + \frac{N_m^2 g}{4} \sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{N_v} \operatorname{Tr} \left(\bar{\Psi}_{\mu_1} \Psi_{\mu_2} \bar{\Psi}_{\mu_3} \Psi_{\mu_4} \right) \operatorname{Tr} \left(\Psi_{\mu_4} \bar{\Psi}_{\mu_3} \Psi_{\mu_2} \bar{\Psi}_{\mu_1} \right) . \tag{5.2}$$

To obtain the full Green functions, consider the trivial equation

$$\lim_{N_m \to \infty} \lim_{N_v \to \infty} \frac{1}{N_m^2 \sqrt{N_v} Z'_{N_m,N_v}} \sum_{i,j=1}^{N_m} \sum_{\beta=1}^{N_v} \int d\Psi_1 d\bar{\Psi}_1 d\Psi_2 d\bar{\Psi}_2 \cdots d\Psi_{N_v} d\bar{\Psi}_{N_v} \times \\ \times \frac{\partial}{\partial \bar{\Psi}_{\beta ij}} \left\{ \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{2n-1}}^{N_v} \left(\bar{\Psi}_{\alpha_{2n-1}} \Psi_{\alpha_{2n-2}} \cdots \bar{\Psi}_{\alpha_1} \bar{\Psi}_{\alpha_1} \Psi_{\alpha_2} \cdots \bar{\Psi}_{\alpha_{2n-1}} \Psi_\beta \right)_{ij} \exp S' \right\} = 0$$

for any positive integer n. After some manipulations, this equation leads to the Schwinger-Dyson equation

$$\sum_{k=1}^{n} \tilde{\phi}(n-k) G'(k-1) + (1+c) G'(n) + g G'(n+1) = 0 \,, \tag{5.3}$$

where n is any positive integer and $\tilde{\phi}(n)$ was defined in eq. (2.3) again with

$$V(M) = \frac{1}{2} M^2 - \frac{g}{4} M^4 \,.$$

JHEP06(2003)044

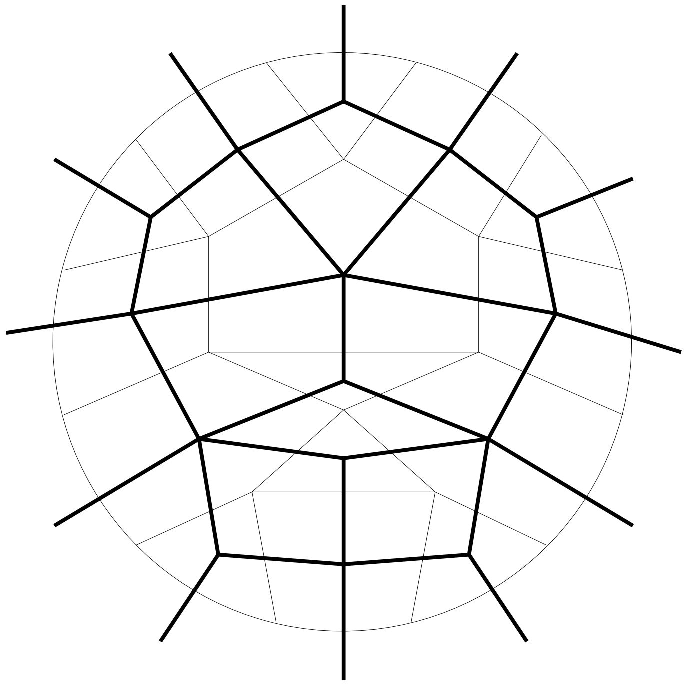


Figure 8: A discretized random orbifold surface with a boundary generated by the fermionic matrix-vector model defined in eq. (5.2). This surface respects reflection symmetry instead of parity. the "spokes" protruding from the great circle converge to the same point on a spherical surface.

Introduce the following generating function of the moments:

$$\omega_1'(z) := \sum_{n=0}^{\infty} G'(n) z^{2n} .$$

Then we may rewrite eq. (5.3) as

$$\omega_1'(z) = \frac{g + [1 + c + gG'(1)]\, z^2}{g + (1+c)z^2 + z^4 \phi(z)} ,$$

where $\phi(z)$ was defined in eqs. (4.5) and (4.6). Hence,

$$\omega_1'(z) = \frac{P_1(z^2) \left[\frac{g}{2} + \left(\frac{3}{2} + c \right) z^2 - \tilde{\rho}(z^2) \right]}{P_3(z^2)} , \tag{5.4}$$

where

$$\begin{aligned} P_1(\zeta) &:= g + \left[1 + c + gG'(1) \right] \zeta , \\ \tilde{\rho}(\zeta) &:= \left[\frac{g}{2} + \left(g\gamma^2 - \frac{1}{2} \right) \zeta \right] \sqrt{1 - 4\gamma^2 \zeta} , \end{aligned}$$

and

$$P_3(\zeta) := \zeta \left[(2+c)g + (2 + 3c + c^2 - g)\zeta + \left(-\frac{4}{3} g\gamma^2 + \frac{1}{9}\gamma^2 + \frac{8}{9} \right) \zeta^2 \right] .$$

Zero and one of the non-zero roots of the cubic polynomial $P_3(\zeta)$ are roots of $\tilde{\rho}(\zeta)$, too. Hence $G'(1)$ is fixed by the requirement that $P_1(\zeta)$ vanish if ζ is equal to the other non-zero root of $P_3(\zeta)$. Then we may read off the full Green functions from $\omega_1'(z)$.

At the critical point, the zeros of

$$\rho'(z) := -\frac{P_1(z^2)\tilde{\rho}(z^2)z}{P_3(z^2)}$$

coalesce with an end-point of its branch cut. Thus the critical values of g, c, and z are

$$g_* = \frac{1}{12},\ c_* = -\frac{11}{6}, \qquad \text{and} \qquad z_* = \frac{1}{2\sqrt{2}}.$$

Near the critical point,

$$g = g_* \exp(-a^2\Lambda_s), \qquad c = c_* \exp(-a^2\Lambda_c), \qquad \text{and} \qquad z = z_* \exp(-a\Lambda_l),$$

where a is small at the continuum limit. Then

$$\rho'(z) \simeq -\frac{1}{4\sqrt{2}} w'(\Lambda_s, \Lambda_l)\sqrt{a},$$

where

$$w'(\Lambda_s, \Lambda_l) := \frac{1}{\Lambda_l}\left(\sqrt{\Lambda_s} - 4\Lambda_l\right)\left(\sqrt{\Lambda_s} + 2\Lambda_l\right)^{\frac{1}{2}}$$

is the renormalised tree-level one-boundary amplitude of this orbifolded string worldsheet. The string susceptibility is defined as in eq. (4.11) with S replaced with S'. It turns out that

$$\chi = \frac{\partial}{\partial g}\left(\tilde{\phi}(2)\right) \simeq (g - g_*)^{1/2}.$$

Hence we conclude again from eqs. (4.5) and (4.6) that

$$\gamma_{\text{str}} = -\frac{1}{2}.$$

The critical exponent is identical to but the tree-level one-boundary amplitude is different from that of ordinary quantum gravity.

6. Conclusion and outlook

It is clear from the above calculation that fermionic matrix-vector models may be used to analyse the behavior of different quantum orbifold geometry. They have the same critical exponent as that of ordinary quantum gravity, but the one-boundary amplitudes are already different at the tree level. It is clearly of interest to study the multi-loop amplitudes and double-scaling or even triple-scaling limit of these models as well as their multicritical generalisations and explore the subsequent ramifications in quantum gravity and string theory.

JHEP06(2003)044

On the more mathematical side, clarification of the relationship between the set of fermionic matrix-vectors of infinite order with the ordered pair of a C* algebra and a vector space, constituents of a noncommutative probability theory of type B as defined in ref. [16], will be desirable. Fermionic matrix-vector models may even be useful in the applications of noncommutative probability of type B to algebraic analysis. (c.f. the application of ordinary random matrix models, the embodiment of noncommutative probability of type A, to the analysis of von Neumann algebras [29].) Furthermore, just like it is interesting to construct rigorous topological or combinatorial arguments to relate the Feynman diagrams of ordinary matrix models with quadrangulated random surfaces [30, 31], it would be desirable to develop a rigorous proof for the identification of the dual of Feynman diagrams of fermionic matrix-vector models with quadrangulated orbifold surfaces and explore the consequences.

Acknowledgments

I am grateful to A. Nica and S.G. Rajeev for providing me valuable insights into the subject matter of this article. I would also like to thank the referee for helpful suggestions for the revision of this article.

References

[1] D.V. Voiculescu, K.J. Dykema and A. Nica, *Free random variables*, *CRM Monograph Series* **1**, American Mathematical Society, 1992.

[2] E. Witten, *The 1/N expansion in atomic and particle physics*, in *Recent developments in gauge theories*, G. 't Hooft et al. eds., Plenum Press, 1980, pp. 403–420.

[3] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[4] F. David, *Planar diagrams, two-dimensional lattice gravity and surface models*, *Nucl. Phys.* **B 257** (1985) 45.

[5] V.A. Kazakov, A.A. Migdal and I.K. Kostov, *Critical properties of randomly triangulated planar random surfaces*, *Phys. Lett.* **B 157** (1985) 295.

[6] J. Ambjørn, B. Durhuus and J. Fröhlich, *Diseases of triangulated random surface models, and possible cures*, *Nucl. Phys.* **B 257** (1985) 433.

[7] M.R. Douglas, *Stochastic master fields*, *Phys. Lett.* **B 344** (1995) 117 [hep-th/9411025].

[8] R. Gopakumar and D.J. Gross, *Mastering the master field*, *Nucl. Phys.* **B 451** (1995) 379 [hep-th/9411021].

[9] M.R. Douglas and M. Li, *Free variables and the two matrix model*, *Phys. Lett.* **B 348** (1995) 360 [hep-th/9412203].

[10] L. Accardi, I.Y. Aref'eva, S.V. Kozyrev and I.V. Volovich, *The master field for large-N matrix models and quantum groups*, *Mod. Phys. Lett.* **A 10** (1995) 2323 [hep-th/9503041].

[11] M. Engelhardt, *Nonrelativistic particle in free random gauge background*, *Nucl. Phys.* **B 481** (1996) 479 [hep-th/9604012].

[12] A. Agarwal, L. Akant, G.S. Krishnaswami and S.G. Rajeev, *Collective potential for large-N hamiltonian matrix models and free Fisher information*, *Int. J. Mod. Phys.* **A 18** (2003) 917 [hep-th/0207200].

[13] R. Speicher, *Multiplicative functions on the lattice of noncrossing partitions and free convolution*, *Math. Ann.* **298** (1994) 611.

[14] G. Kreweras, *Sur les partitions non-croisées d'un cycle*, *Discrete Math.* **1** (1972) 333.

[15] V. Reiner, *Non-crossing partitions for classical reflection groups*, *Discrete Math.* **177** (1997) 195.

[16] P. Biane, F. Goodman and A. Nica, *Non-crossing cumulants of type B*, math.OA/0206167.

[17] Y. Makeenko and K. Zarembo, *Adjoint fermion matrix models*, *Nucl. Phys.* **B 422** (1994) 237 [hep-th/9309012].

[18] J. Ambjørn, C.F. Kristjansen and Y. Makeenko, *Generalized Penner models to all genera*, *Phys. Rev.* **D 50** (1994) 5193 [hep-th/9403024].

[19] N. Marshall, G.W. Semenoff and R.J. Szabo, *Critical behavior of a fermionic random matrix model at large-N*, *Phys. Lett.* **B 351** (1995) 153 [hep-th/9410214].

[20] G.W. Semenoff and R.J. Szabo, *Fermionic matrix models*, *Int. J. Mod. Phys.* **A 12** (1997) 2135 [hep-th/9605140].

[21] V.A. Kazakov and A.A. Migdal, *Induced QCD at large-N*, *Nucl. Phys.* **B 397** (1993) 214 [hep-th/9206015].

[22] F.A. Brezin, *The method of second quantization*, *Pure and Applied Physics* **24**, Academic Press, 1966.

[23] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *2-d gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153].

[24] J. Ambjørn, B. Durhuus and T. Jónsson, *Quantum geometry*, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, 1997.

[25] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[26] S.R. Wadia, *On the Dyson-Schwinger equations approach to the large-N limit: model systems and string representation of Yang-Mills theory*, *Phys. Rev.* **D 24** (1981) 970.

[27] F. David, *Loop equations and nonperturbative effects in two-dimensional quantum gravity*, *Mod. Phys. Lett.* **A 5** (1990) 1019.

[28] M. Staudacher, *Combinatorial solution of the two matrix model*, *Phys. Lett.* **B 305** (1993) 332 [hep-th/9301038].

[29] D.V. Voiculescu, *Free entropy*, *Bull. London Math. Soc.* **34** (2002) 257 [math.OA/0103168].

[30] D. Bessis, C. Itzykson and J. B. Zuber, *Quantum field theory techniques in graphical enumeration*, *Adv. in Appl. Math.* **1** (1980) 109.

[31] D.M. Jackson, *On an integral representation for the genus series for 2-cell embeddings*, *Trans. Amer. Math. Soc.* **344** (1994) 755.